CRITICAL REMARKS ON THE APPLICATION OF
THE YIELD LINE METHOD ON MASONRY [a]

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ABSTRACT

The present paper focuses on the application of the yield line method to estimate the pressure bearing capacity of laterally loaded masonry walls. The characteristics of the function of bending moment coefficient are given first to assess the results of numerical analysis and the proposed methods in literature. The shortcomings of applying the yield line method on masonry are discussed and a proposal for the possible modifications is given, based on the length of the first crack. The numerical simulation has been performed to follow the progression of cracks up to the point of collapse. Primarily, two kinds of masonry walls have been considered through the study: (a) Masonry walls which show no crack through the units, and (b) Masonry walls which show cracks through the units. The numerical results have been compared with the proposed modification on yield line method to give a clear explanation of the behaviour and to determine the critical factors that influence the pressure bearing capacity.

KEYWORDS: bending moment coefficient, critical aspect ratio, lateral loading of masonry, numerical simulation, out-of-plane failure, yield line method

INTRODUCTION

The behaviour of two-way bending of masonry walls is one of the most challenging and ill-understood problems in the field of masonry. Although several attempts have been devoted to understand out-of-plane behaviour and to estimate the pressure bearing capacity, the state of art in the field is still unsatisfactory and many problems have to be explained and clarified.

Several experimental studies on laterally loaded masonry walls have been reported in literature Johansen [1], Haseltine [2], Hendry [3], Hendry et al. [4], West et al. [5], Lawrence [6], Baker [7] and Van der Pluijm [8]. The studies have been showed that the development of crack pattern at failure is similar to yield line pattern in reinforced concrete slabs. Some empirical formulas are proposed to estimate the pressure capacity, but there is no general and clear explanation for the obtained results.

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[a] Dedicated to Prof. Dr.-Ing. Habil. I.R. Heinz Müller on the occasion of his 80th birthday, holder of the Chair of Structural Mechanics/Statics at Dresden University of Technology from 1970 to 1996.

The similarity of the failure pattern in masonry walls and reinforced concrete slabs has been driven to apply Johansen's yield line method to laterally loaded masonry walls. Haseltine [2] and Anderson [9] proposed to calculate pressure bearing capacity by yield line method, in which the orthogonal ratio μ, is taken as being equal to the flexural strength ratio of the masonry, provided that, flexural strength values are taken from wallette test as established by West et al. [5]. The yield line method has been first introduced in form of tabulated moment coefficients, for the purpose of friendly design procedure in British standard BS 5628 [10], and later in Eurocode 6 [11].

The following formulas are given in Eurocode 6 to calculate the moment resistance of masonry walls subjected to uniform pressure W:

$$M_1 = \alpha_1 W l^2 \quad \text{and} \quad M_2 = \alpha_2 W l^2 \qquad \text{where } \alpha_1 = \mu \cdot \alpha_2 \tag{1}$$

Where M_1, M_2 are the moment resistances in directions perpendicular and parallel to bed joints respectively. The bending coefficient α_2 is a function of the flexural orthotropy ratio μ and the aspect ratio $r = h/l$ and given in tabulated form for each support condition. The flexural orthotropy ratio given as

$$\mu = \frac{f_{xk1}}{f_{xk2}} \tag{2}$$

where f_{xk1} the characteristic flexural strength of masonry when the plane of failure parallel to the bed joints and f_{xk2} the characteristic flexural strength of masonry when the plane of failure perpendicular to the bed joints.

Later on, many investigations have confirmed that the yield line method is unsafe for most cases and overestimates the pressure bearing capacity. So far, there is no theoretical justification for applying the method to a brittle material. One modification introduced by Sinha [12] to account the orthotropic stiffness which enhanced the correlation with the experimental results.

The test observations reported by Baker et al. 2005; Drysdale and Hamid, 2005, (referenced in Maluf et al.[13]) regarding neglecting the moment at the first crack, have led to some modifications on yield line method, which have introduced in Canadian Code CAN-CSA S304.1-04 [14]. However, the principle which used to find the moment coefficients is underestimating the capacity for aspect ratios far from one even beyond the moment coefficients of one way bending. This is a result of ignoring the dissipated energy along the first crack.

According to limit load theory it is necessary to determine an upper (kinematic) and a lower (static) value of the ultimate load (s. [17]). If a load can be found which is safe and statically permitted and the number of full plastic moments (or line moments) leads to a failure chain, than the determined limit load is the actual one ([17]). The assumed failure chain is that which really will occur. How it was shown in [17] (p. 229, tab. 6.2) for an isotropic, simply supported plate, the lower bound and the upper bound falling together in case of easy yield lines or they are lying very narrow to each other. Here the exact determination of the yield or failure line figure can be done by the help of the equilibrium equations and the application of the maximum principle (see [19], p. 30). That can be assumed in case of lateral loaded masonry panels.

The intention of the present paper is to verify the capability of the yield line method to estimate the pressure bearing capacity of masonry walls as well as to give a clear description for two bending behaviour of masonry walls. The investigation is performed on four side supported walls (case E in Eurocode 6), where the principles and the methodology can be extended to the other support conditions.

PROPERTIES OF α_2 FUNCTION

To assess the results of this study and the methods proposed in literature, the characteristics of the function α_2 are considered.

By assuming two masonry walls P_1 and P_2 of dimensions $l \times h_1$ and $l \times h_2$ and have the same units and joint properties, which indicate the equality of flexural orthotropy for both walls, Figure 1:

If $h_1 < h_2$ then the aspect ratio $r(P_1) < r(P_2)$ and the area of the wall $A(P_1) < A(P_2)$ $\hspace{2em}$ (3)

Since both walls have the same length l, the pressure bearing capacity of wall P_1 should be not less than that of wall P_2:

$$W(P_1) \geq W(P_2)$$ $\hspace{2em}$ (4)

The moment resistance for both masonry walls are:

$$M_2(P_1) = \alpha_2(P_1) \cdot W(P_1) \cdot l^2 \hspace{2em} M_2(P_2) = \alpha_2(P_2) \cdot W(P_2) \cdot l^2$$ $\hspace{2em}$ (5)

P_1 and P_2 built from the same material and have the same thickness. The moment resistances therefore are equal $M_2(P_1) = M_2(P_2)$. This gives:

$$\alpha_2(P_1) \cdot W(P_1) \cdot l^2 = \alpha_2(P_2) \cdot W(P_2) \cdot l^2$$ $\hspace{2em}$ (6)

By considering the inequality in pressure bearing capacity (4) then

$$\alpha_2(P_1) \leq \alpha_2(P_2)$$ $\hspace{2em}$ (7)

Mathematically this can be written as

$$\left. \begin{array}{l} \forall P_1, P_2 \in \alpha_2(\mu = const) \\ r(P_1) < r(P_2) \end{array} \right) \Rightarrow \alpha_2(P_1) \leq \alpha_2(P_2)$$ $\hspace{2em}$ (8)

The above expression means that, the function α_2 is *monotonically non-decreasing* for constant flexural orthotropy.
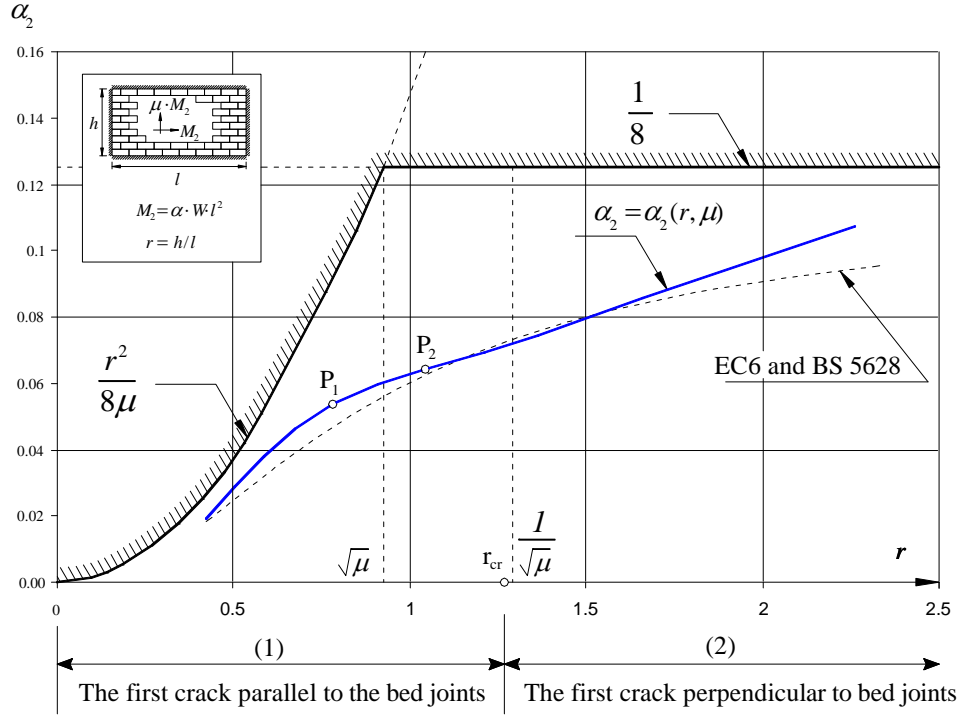
Figure 1 α_2 bending coefficient function with the upper bound

The upper bound of α_2 coefficient can be found by considering the moment coefficients of one-way bending, in directions parallel and perpendicular to bed joints.
In direction parallel to bed joints

$$M_2 = \alpha_2 W l^2 \leq \frac{W l^2}{8} \tag{9}$$

$$\frac{M_2}{W l^2} = \alpha_2 \leq \frac{1}{8} \tag{10}$$

In direction perpendicular to bed joints

$$M_1 = \alpha_1 W l^2 \leq \frac{W h^2}{8} \tag{11}$$

$$\alpha_1 \leq \frac{r^2}{8} \text{ or } \alpha_2 \leq \frac{r^2}{8\mu} \tag{12}$$

Consequently, the upper bound of α_2 values is

$$\alpha_2 \leq \begin{cases} \dfrac{1}{8} \\[2mm] \dfrac{r^2}{8\mu} \end{cases} \tag{13}$$

THE CRITICAL ASPECT RATIO

There is a point on the curve of α_2 function, at which no first horizontal or vertical crack would appear. The aspect ratio at this point can be defined as the *critical aspect ratio r_{cr}*. For masonry walls which have aspect ratio equal to the critical aspect ratio, the cracks propagate simultaneously in both directions, which would result in form of diagonal cracks. The value of the critical aspect ratio for walls of equally flexural strengths and equally stiffness is equal to 1. However, for most masonry walls, the strength and the stiffness in direction parallel to bed joints are higher than that in direction perpendicular to head joints. This could shift the value of the critical aspect ratio far from 1 as the orthotropic properties are varying.

SHORTCOMINGS OF THE YIELD LINE METHOD & POSSIBLE MODIFICATIONS

Yield line method as first proposed by Johansen [1], is based on the work-equilibrium, where the internal virtual work equals the external virtual work. It is needed to calculate several crack patterns and to use the lowest value of collapse pressure.

Yield line method supposes equal moment resistances over yield lines prior to collapse. However, for brittle or quasi-brittle materials, all possible lines of failure are experienced some damage prior to collapse. The moment resistances along the yield lines are not equal, and they change as the cracks propagate.

For possible use of yield line method, it can be applied over several phases of crack propagation. Each phase needs to calculate the dissipated energy and the work applied by external pressure. However, to simplify the calculation scheme only two phases are considered:

1- The propagation of the first horizontal (vertical) crack,
2- The propagation of the diagonal cracks.

To apply the yield line method, it would be feasible to focus on the phase of failure, when the cracks are appearing simultaneously in both directions, parallel and perpendicular to bed joints (or as diagonal cracks). Therefore, the yield line method can be applied in the second phase, and here there are principally two aspects affect the results:

1- The moment resistance over the yield line are reduced due to damage, this can be included in the formulation by introducing a reduction factor κ into the moment resistance at the first crack.
2- Due to that damage, the energy produced by the applied pressure is also reduced after dissipation, however such factor can be ignored if the length of the first crack is small in comparison to the diagonal cracks. The ignoring of this factor would result in underestimation of the pressure capacity for aspect ratios far from one. The error resulted by ignoring this factor can be observed in Canadian code approach.

ADOPTION OF THE YIELD LINE METHOD TO MASONRY

In order to get an explanation of two-way bending behaviour of masonry based on the first crack length, the first aspect is only implemented in the following formulation. The obtained results have maintained by employing the properties of α_2 function.
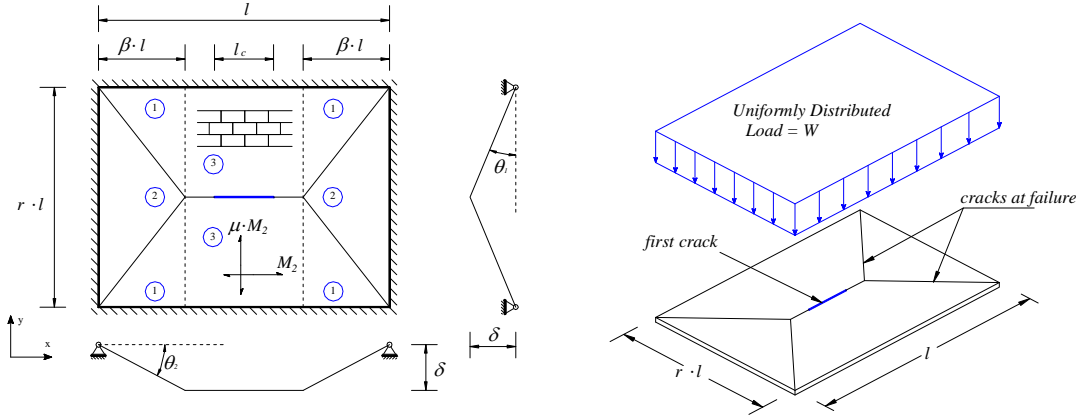
Figure 2 **The calculation scheme for laterally loaded masonry walls based on the modified yield line method**

At failure the potential energy E exposed by loads moving must equal the energy dissipated in yield lines rotating D, this states:

$$E = D \tag{14}$$

The potential energy E exposed by loads moving is:

$$E = \frac{r \cdot W \cdot l^2}{6}(3 - 2 \cdot \beta) \tag{15}$$

The dissipation of energy within the yield lines can be calculated by introducing a reduction factor κ into the first crack energy, so that, the length at which the moment resistance vanishes is $l_c = \kappa \cdot (1 - 2\beta) \cdot l$

$$D = 2 \cdot [2\beta + \kappa(1 - 2\beta)] \cdot l \cdot \mu \cdot M_2 \cdot \theta_1 + 2 \cdot M_2 \cdot r \cdot l \cdot \theta_2 \tag{16}$$

By substituting the rotation angles

$$\theta_1 = \frac{2}{r \cdot l} \qquad \theta_2 = \frac{1}{\beta \cdot l} \tag{17}$$

$$D = \frac{M_2}{\beta \cdot r} \cdot \left[8(1 - \kappa) \cdot \mu \cdot \beta^2 + 4 \cdot \kappa \cdot \mu \cdot \beta + 2r^2 \right] \tag{18}$$

By equating the potential energy E and the internal dissipated energy D gives

$$\frac{M_2}{w \cdot l^2} \cdot = \frac{r^2}{12} \cdot \frac{3\beta - 2 \cdot \beta^2}{4(1 - \kappa) \cdot \mu \cdot \beta^2 + 2 \cdot \kappa \cdot \mu \cdot \beta + r^2} \tag{19}$$

Thus,

$$\alpha_2 \cdot = \frac{r^2}{12} \cdot \frac{3\beta - 2 \cdot \beta^2}{4(1-\kappa) \cdot \mu \cdot \beta^2 + 2 \cdot \kappa \cdot \mu \cdot \beta + r^2} \le \frac{r^2}{8\mu} \tag{20}$$

the value of β should be determined for minimum pressure, or

$$\frac{\partial \alpha_2}{\partial \beta} = 0 \quad \text{gives} \quad 4 \cdot \mu(3 - 2 \cdot \kappa) \cdot \beta^2 + 4r^2\beta - 3r^2 = 0 \tag{21}$$

Solving for β

$$\beta = \frac{r^2}{2\mu(3-2\kappa)}\left[-1 + \sqrt{1 + \frac{3\mu}{r^2} \cdot (3-2\kappa)}\right] \le \frac{1}{2} \tag{22}$$

The same procedure can be applied to derive the equations for aspect ratios higher than r_{cr}

$$\alpha_2 = \frac{r^2}{12} \cdot \frac{3\beta - 2 \cdot \beta^2}{4(1-\kappa) \cdot \beta^2 \cdot r^2 + 2 \cdot \kappa \cdot \beta \cdot r^2 + \mu} \le \frac{1}{8} \tag{23}$$

$$\beta = \frac{\mu}{2r^2(3-2\kappa)}\left[-1 + \sqrt{1 + \frac{3r^2}{\mu}(3-2\kappa)}\right] \le \frac{1}{2} \tag{24}$$

When $\kappa = 1 \Rightarrow \alpha_2$ is identical with EC6 and BS 5628 which are based on traditional yield line method. When $\kappa = 0 \Rightarrow \alpha_2$ is identical with Canadian Code CAN-CSA S304.1-04 [14], which is based on zero-moment assumption at the first crack.

NUMERICAL SIMULATION
The intention of numerical simulation is to follow the progression of cracks up to the point of collapse. An accurate procedure has been employed for modelling based on the results of Bakeer [15] which developed for collapse analysis of masonry structures. Mainly two kinds of masonry walls are considered through the numerical simulation: (a) Masonry walls which show cracks through the units. The material of masonry units is brittle (eg. walls of Autoclaved Aerated concrete units). A deletion of the failed finite elements was performed, to visualize the propagation of cracks, and (b) Masonry walls which show no cracks through the units. The material of masonry units has high tensile strength and the failure occurs only at the joints (e.g. walls from Calcium silicate units)
The modelling of masonry walls have been carried out using LS-DYNA software and an explicit transient analysis has been performed. The pressure has been applied increasingly up to collapse, so that, the post failure behaviour can be captured after initial cracking. The principles can be found in Bakeer [15], however, what is necessary to focus on in this context are the crack patterns obtained by numerical analysis. The capability of the numerical tool has been verified with the available empirical data on lateral loading of masonry, Figure 3.
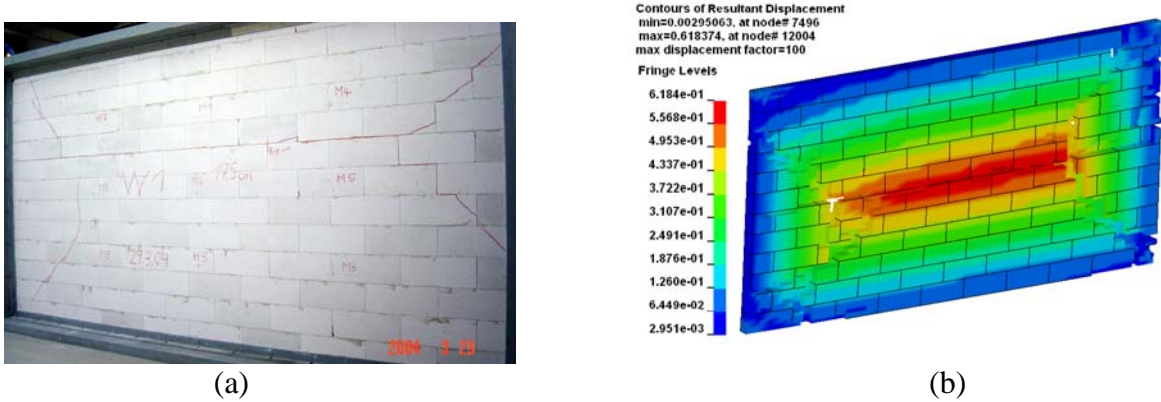
| (a) | (b) |

Figure 3   Crack pattern of autoclaved aerated concrete masonry wall with dimensions 6 x 3 x 0.175 m, (a) the tested wall, Jäger et al. [16], (b) the numerical simulation

EXPLANATION OF NUMERICAL RESULTS BASED ON κ FACTORS

At the beginning of loading the distribution of moments within the wall depends on the aspect ratio and the stiffness orthotropy only, but not on the flexural orthotropy ratio μ.

The moment ratio M_1 / M_2 at the beginning of loading in case of isostiffness is equal to $1/r^2$. The wall, therefore, starts to develop moments in the 1st direction higher than that in the 2nd direction. In masonry walls which have $\mu \leq 1$ the moment resistant in 1st direction less than the moment resistant in 2nd direction. This leads to develop the first crack along the bed joints. The progression of the first crack increases until the moment ratio M_1 / M_2 reaches the value of μ.
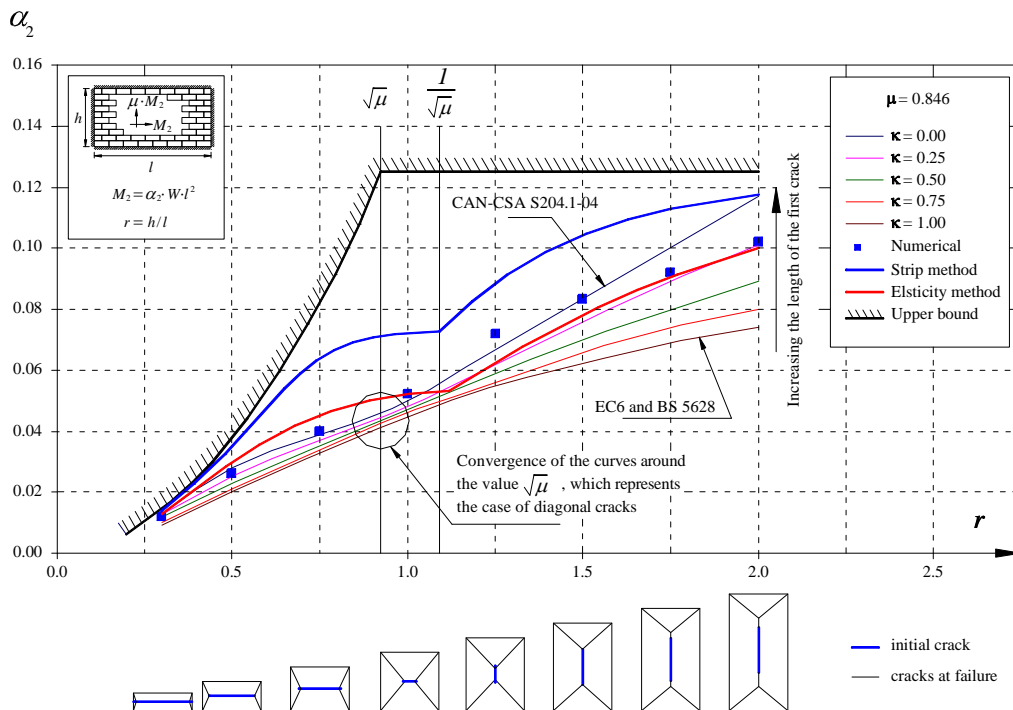

Figure 4   Comparison of numerical results based on κ factor for $\mu = 0.846$

Developing of cracks, zero-moment at the first cracked line

Aspect ratio of the wall =2

Developing of cracks, zero-moment at the first cracked line
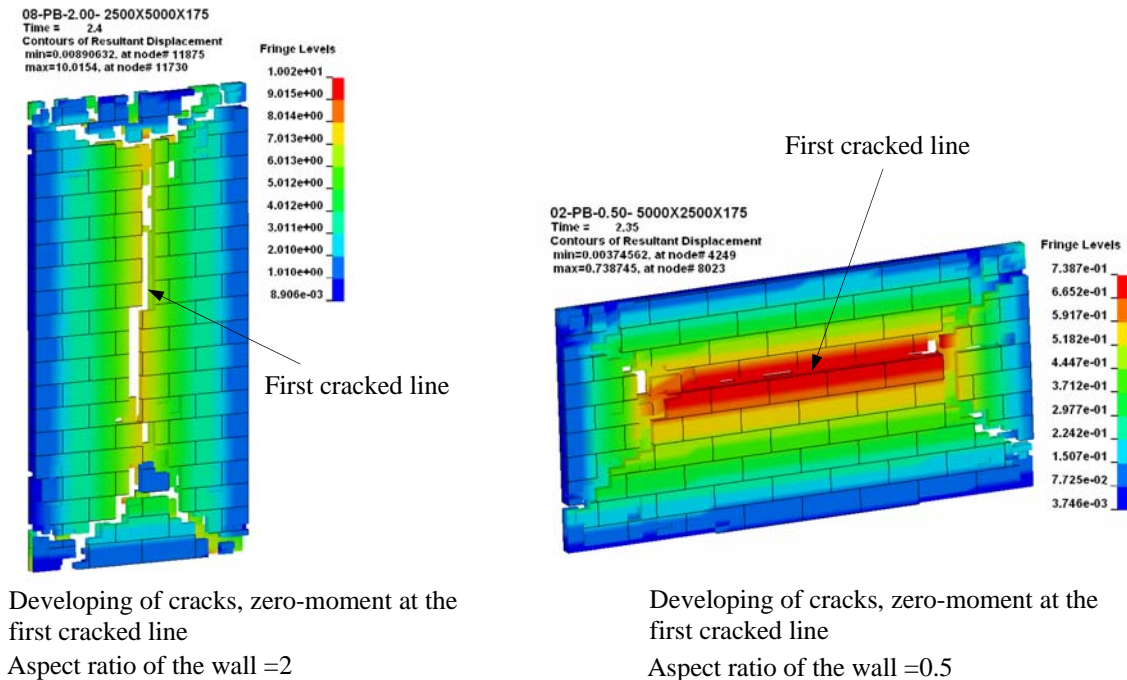
Aspect ratio of the wall =0.5

Figure 5   Crack patters of aerated concrete masonry walls, the zero moment at the first crack is observed in both cases when the first crack parallel or perpendicular to the bed joints.

The results of numerical simulation can be classified into two groups:

(a) Masonry walls which show failure in units (e.g. autoclaved aerated concrete units)
At the beginning of loading, an initial crack has been observed for all aspect ratios, and the pressure bearing capacity has been fitted well with the curve ($\kappa = 0$). The crack propagation through the units in all directions and for all aspect ratios, make masonry wall behave similar to a slab of homogenous material, Figure 4 and Figure 5. The numerical results also are close to the solution provided by elasticity theory, as a result of the brittle failure. The brittle failure can be achieved by very small increase of the applied pressure after the initial cracking. Therefore, it is reasonable to consider the pressure at the initial crack as the pressure bearing capacity of the wall, which can be calculated from elasticity theory.

(b) Masonry walls which show no failure in units (e.g. calcium silicate units)
Several walls with flexural orthotropy ratios 0.745, 0.622 and 0.257 are considered. The comparisons of the numerical results with the results of different methods are presented in Figures 7, 8 and 9. The first crack observed clearly along the bed joints at the middle of the wall for long walls, but for high walls there is no possibility for the first crack to be propagated through the units Figure 6.
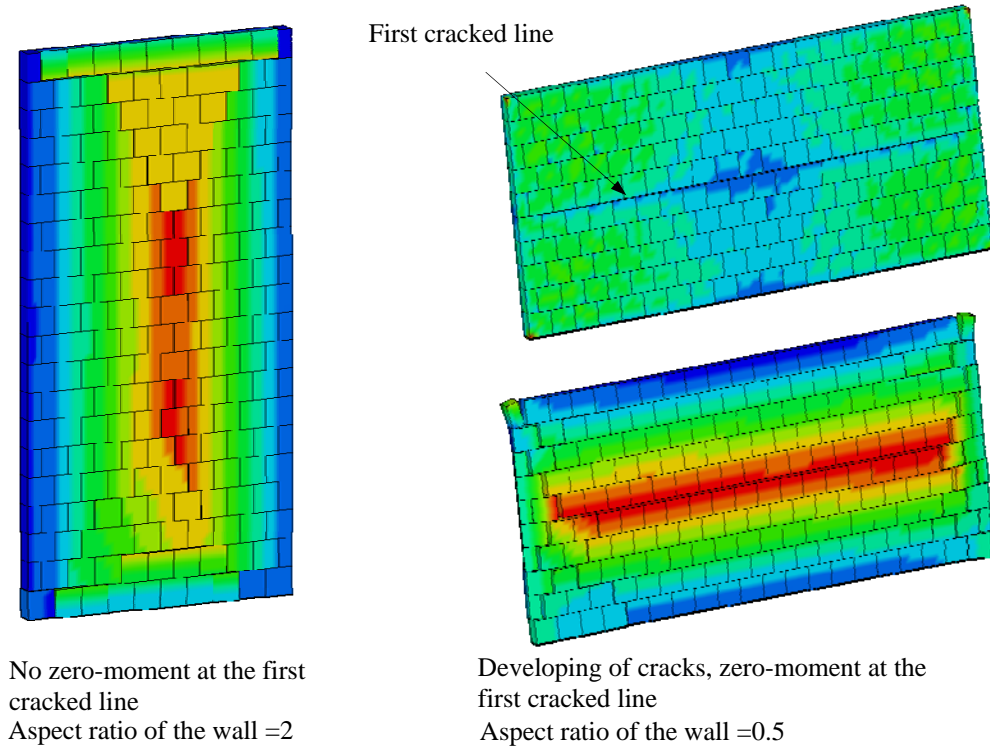
First cracked line

No zero-moment at the first cracked line
Aspect ratio of the wall =2

Developing of cracks, zero-moment at the first cracked line
Aspect ratio of the wall =0.5

Figure 6   The mechanism of failure for Calcium silicate walls

The factor κ gives good representation for the length of the first crack, for aspect ratios less than r_{cr} the numerical results tend to be up or close to the curve $\kappa = 0$, and for aspect ratios higher than r_{cr} the numerical results tend to be between the curves $\kappa = 1$ and $\kappa = 0.5$.

So far, modified yield line method considers only the flexural properties of the material. However, the stiffness orthotropy K has significant influence on the pressure bearing capacity, especially for small flexural orthotropy. Another shortcoming can be recognized when the curves of the modified yield line method cuts the upper bound of α_2 function. As mentioned before, this has caused by ignoring the term of energy dissipated at the first crack.

The curves which represent the elasticity solutions in Figures 7, 8 and 9 show good agreement with the numerical results but they do not meet the conditions of monotonically non-decreasing property and the upper bound of α_2 function, particularly for small flexural orthotropy. The elasticity results are obtained for isostiffness walls, which is indeed not true for masonry. Masonry walls which have higher flexural strength in some direction are always accompanied with higher stiffness in that direction. By including the stiffness orthotropy K in elasticity method, the results can meet the conditions of α_2 function.

The yield line method gives critical aspect ratio of $\sqrt{\mu}$ while the strip method gives critical aspect ratio of $r_{cr} = \sqrt{\dfrac{K}{\mu}}$. For elasticity theory Poisson ratio has an effect on this value.
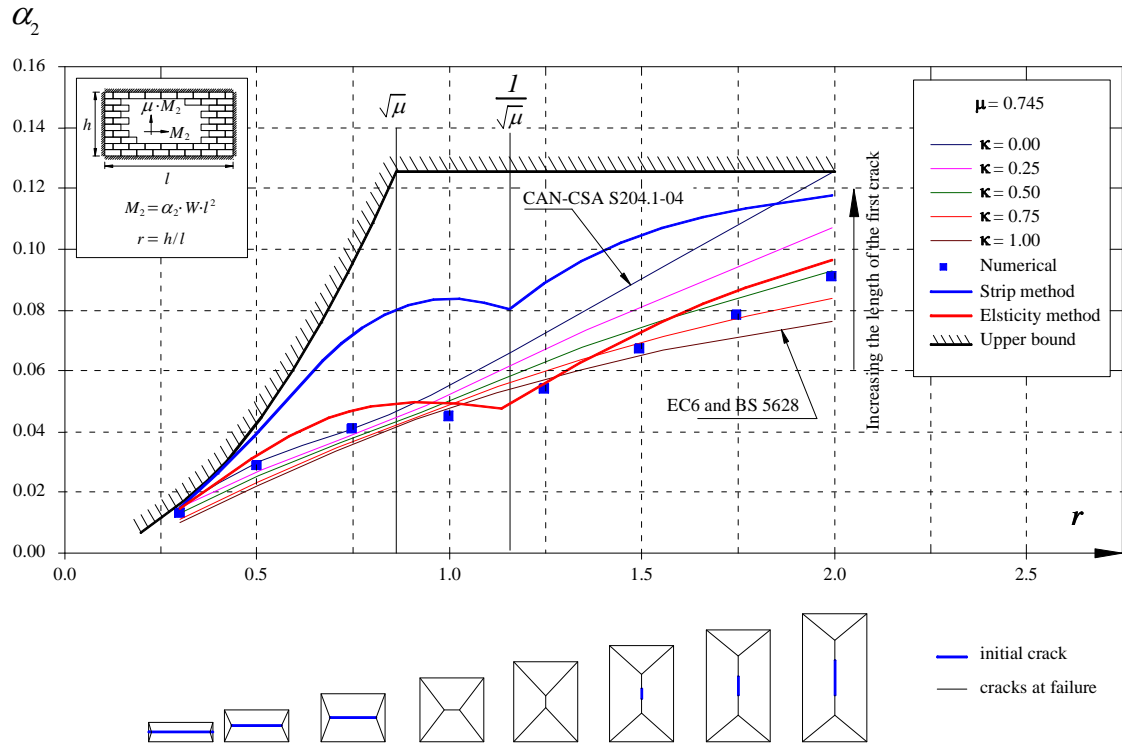
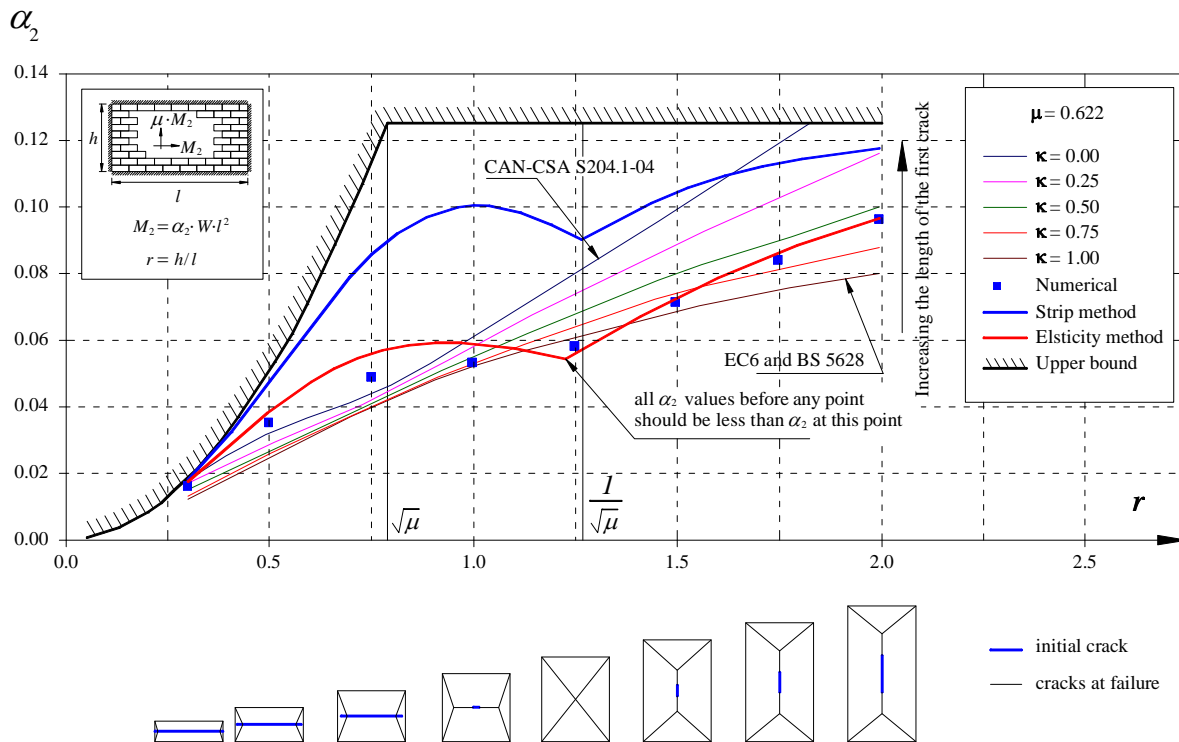**Figure 7   Comparison of numerical results based on κ factor for μ = 0.745**

Figure 8   Comparison of numerical results based on κ factor for μ = 0.622
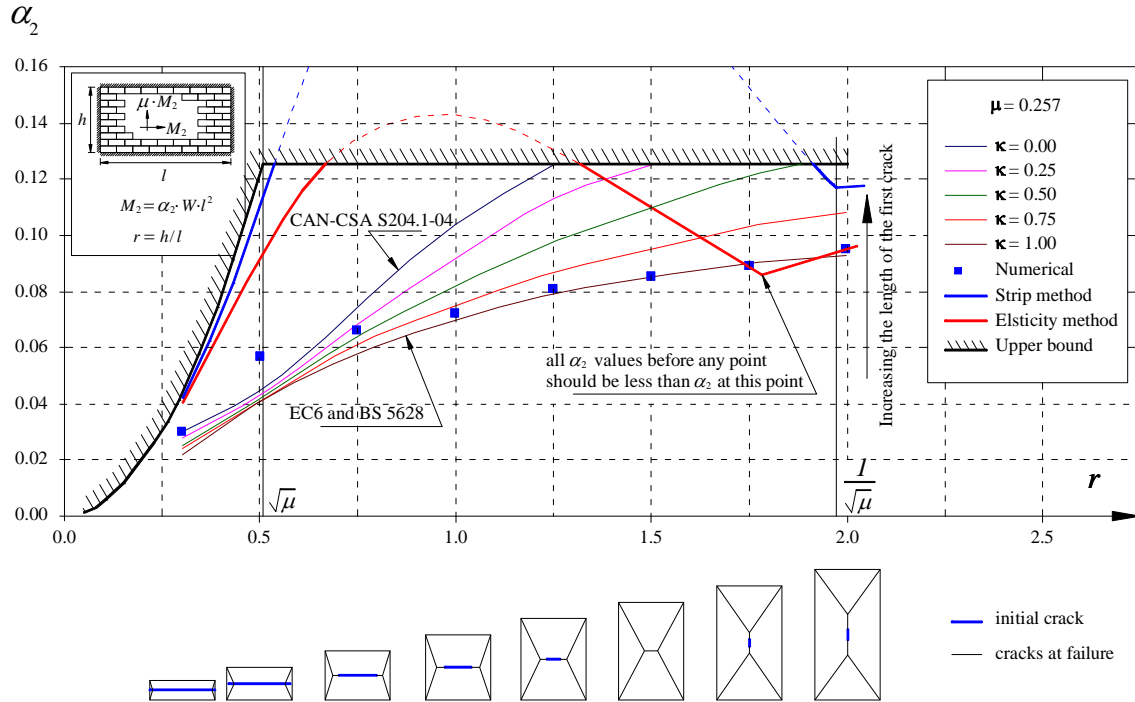
Figure 9   Comparison of numerical results based on κ factor for μ = 0.257

CONCLUDING REMARKS AND FUTURE RESEARCH ASPECTS

Masonry is quasi-brittle material. Beside the results of the present paper, several remarks in literature mentioned that yield line method is unsafe for most cases to be applied for masonry.

The proposed modified yield line method in this paper is based on the length of the first crack. In Canadian standard, the length of the first crack has been handled in the same way for long walls and high walls. However, the calculation of the first crack length is related basically on the aspect ratio of the wall, the flexural orthotropy and stiffness orthotropy. The introducing of κ factor into the first crack length has enabled to explain the crack patterns obtained in numerical analysis. The proposed modifications on yield line method are not accurate without including the stiffness orthotropy, and it is recommended to include the stiffness orthotropy in the design procedure in form of μ / K ratio.

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